Rene’ Descartes is commonly credited for devising the **Rational Root Theorem**.

The theorem states: Given a polynomial equation of the form

\[ 0 = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n \]

Any rational root of the polynomial equation must be some integer factor of \( a_n \) divided by some integer factor of \( a_0 \)

Given the following polynomial equations, determine all of the “POTENTIAL” rational roots based on the Rational Root Theorem and then using a synthetic division to verify the most likely roots.

1. \( x^3 + x^2 - 8x - 12 = 0 \)

   **Potential Rational Roots:**
   \[ \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \]

   **Actual Roots:** \( 3, -2, -2 \)

2. \( 4x^3 - 12x^2 + 5x + 6 = 0 \)

   **Potential Rational Roots:**
   \[ \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{4}, \pm \frac{1}{4}, \pm \frac{3}{2} \]

   **Actual Roots:** \( \pm 2, \pm \frac{1}{2}, \pm \frac{3}{2} \)

**The Remainder Theorem** suggests that if a polynomial function \( P(x) \) is divided by a linear factor \( (x - a) \) that the quotient will be a polynomial function, \( Q(x) \), with a possible constant remainder, \( r \), which could be written out as:

\[ P(x) = (x - a) \cdot Q(x) + r \]

*If this seems a little complicated consider a similar statement but just using integers. For example (using the same colors to represent similar parts), \( 70 \div 6 = 11 \) with remainder 4 which could also be rewritten as: \( 70 = 6 \cdot 11 + 4 \)*

The Remainder Theorem also leads to another important idea, **The Factor Theorem**. To state the Factor Theorem, we only need to evaluate \( P(a) \) from the Remainder Theorem.

\[ P(a) = (a - a) \cdot Q(a) + r \]

:Substitute “a” in for each “x”

\[ P(a) = (0) \cdot Q(a) + r \]

:Simplify \((a - a) = 0\)

\[ P(a) = r \]

:Simplify \(0 \cdot Q(a) = 0\)

This is an important fact that basically states the remainder of the statement \( P(x) \div (x - a) \) is \( P(a) \).
Using the Remainder or Factor Theorem answer the following.

3. **Using Synthetic Division evaluate**

\[ x^3 + x^2 - 8x - 12 \text{ when } x = 3 \]

\[ \begin{array}{c|cccc}
3 & 1 & 1 & -8 & -12 \\
\hline
& 3 & 12 & 12 & 0 \\
\end{array} \]

\[ x^3 + x^2 - 8x - 12 = (x-3)(x^2 + 4x + 4) \]

\[ P(3) = (3)^3 + (3)^2 - 8(3) - 12 = 0 \]

4. **Use Synthetic Division to find the remainder**

\[ \text{of } \frac{x^3 + x^2 - 8x - 12}{x-3} \]

\[ \begin{array}{c|cccc}
3 & 1 & 1 & -8 & -12 \\
\hline
& 3 & 12 & 12 & 0 \\
\end{array} \]

5. **Using Synthetic Division evaluate** \( f(-2) \) **given**

\[ f(x) = 3x^4 + 7x^2 - 8x + 12 \]

\[ \begin{array}{c|cccc}
-2 & 3 & 0 & 7 & -8 & 12 \\
\hline
& -6 & 19 & -46 & 104 \\
\end{array} \]

\[ 3x^4 + 7x^2 - 8x + 12 = (x+2)(3x^3 - 6x^2 + 19x - 46) + 104 \]

\[ f(-2) = 3(-2)^4 + 7(-2)^2 - 8(-2) + 12 = 104 \]

6. **Use Synthetic Division to determine the**

**quotient of** \( f(x) \) **and** \( g(x) \), **given**

\[ f(x) = 3x^4 + 7x^2 - 8x + 12 \text{ and } g(x) = x + 2 \]

\[ \begin{array}{c|cccc}
-2 & 3 & 0 & 7 & -8 & 12 \\
\hline
& -6 & 19 & -46 & 104 \\
\end{array} \]

\[ 3x^4 - 6x^2 + 19x - 46 + \frac{104}{x+2} \]

7. **Given** \( f(x) = (x + 5) \cdot Q(x) + 8 \), **evaluate** \( f(-5) \).

\[ f(-5) = (-5 + 5)Q(-5) + 8 \\
= 0 \cdot Q(-5) + 8 \\
= 0 + 8 \\
= 8 \]

8. **Given** \( \frac{f(x)}{x-6} = Q(x) \) **with a remainder 3, evaluate** \( f(6) \).

\[ f(x) = (x-6)Q(x) + 3 \\
= (6-6)Q(6) + 3 \\
= 0 \cdot Q(6) + 3 \\
= 3 \]

9. **Consider** \( f(x) = 2x^3 - x^2 + 3x + 4 \) **and that** \( f(b) = 5 \)

\[ \begin{array}{c|cccc}
b & 2 & -1 & 3 & 4 \\
\hline
& 2 & -5 & b & 5 \\
\end{array} \]

What value should be in box labeled “a”?

10. **Consider** \( g(x) = x^3 + 3x^2 - 2x + 4 \) **and Justin used synthetic division to divide** \( (x^3 + 3x^2 - 2x + 4) \) **by** \( (x - k) \). **His work is partially shown below. Using this information determine** \( f(k) \).

\[ \begin{array}{c|cccc}
k & 1 & 3 & -2 & 4 \\
\hline
k & ka & kb & c & f(k) = C \\
\end{array} \]

\[ \frac{x - k = 0}{+k + k} \]

\[ \frac{x = k}{1 a b c} \]

\[ f(k) = C \]

\[ C \]
Using any available techniques determine the following (find exact answers).

11. Find all of the solutions to the polynomial equation \( x^4 - 3x^3 + 6x^2 - 12x + 8 = 0 \)

12. Find all zeros of the polynomial function \( f(x) = x^4 - 4x^3 + x^2 + 8x - 6 \)